Chii-Huei Yu, International Journal of Computer Science and Mobile Applications,
Vol.1 Issue. 2, August- 2013, pg. 1-8 **ISSN: 2321-8363**

INTERNATIONAL JOURNAL OF COMPUTER SCIENCE AND MOBILE APPLICATIONS - IJCSMA

The Derivatives of Two Types of Functions

Chii-Huei Yu

Department of Management and Information, Nan Jeon Institute of Technology, Tainan City, Taiwan
chiihuei@mail.njtc.edu.tw

Abstract

This paper takes the mathematical software Maple as the auxiliary tool to study the differential problem of two types of functions. We can obtain the Fourier series expansions of any order derivatives of these functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of calculating their higher order derivative values. In addition, we provide two functions to evaluate their any order derivatives, and calculate some higher order derivative values practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords*:* derivatives; Fourier series expansions; differentiation term by term theorem; Leibniz differential rule; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1]-[7].

In calculus courses, finding the n-th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x=c$, in general, needs to go through two procedures: firstly evaluating the n-th order derivative $f^{(n)}(x)$ of $f(x)$, and secondly taking $x=c$ into $f^{(n)}(x)$. These two procedures will make us face with increasingly complex calculations when calculating higher order derivative values of a function (i.e. n is large), Therefore, to obtain the answers by manual calculations is not easy. In this paper, we study the differential problems of the following two types of functions

$$f(x)=e^{ax}\ln[b+c\cos(\lambda x+\beta)] \tag{1}$$

$$g(x)=e^{ax}\ln[b+c\sin(\lambda x+\beta)] \tag{2}$$

where a,b,c,λ,β are real numbers, $\lambda \neq 0$, $b > |c|$. We can obtain the Fourier series expansions of any order derivatives of these two types of functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of evaluating their higher order derivatives values ; these are the major results of this paper (i.e., Theorems 1, 2). As for the study of related differential problems can refer to [8]-[21]. In addition, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and two formulas used in this study.

Notation.

Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, a, b are real numbers. We denote a the real part of z by $\text{Re}(z)$, and b the imaginary part of z by $\text{Im}(z)$.

Formulas.

(i) $\sum_{k=1}^{\infty} \frac{1}{k}(-x)^k = -\ln(1+x)$, where $-1 < x \leq 1$.

(ii) geometric series. $\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k$, where z is a complex number, $|z| < 1$.

Next, we introduce two important theorems used in this paper.

Differentiation term by term theorem ([22]).

If, for all non-negative integer k , the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b) , (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) . Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

Leibniz differential rule ([23]). Let n be a positive integer. If $f(x), g(x)$ are functions such that their m -th derivatives $f^{(m)}(x), g^{(m)}(x)$ exist for all $m = 1,..., n$. Then the n -th derivative of product function $f(x)g(x)$,

$$(fg)^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)}(x) g^{(m)}(x)$$

, where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

Before deriving our major results, we need a lemma.

IJCSMA
Chii-Huei Yu, International Journal of Computer Science and Mobile Applications,
Vol.1 Issue. 2, August- 2013, pg. 1-8 ISSN: 2321-8363

Lemma A. Let b, c, λ, β be real numbers, $\lambda \neq 0, b > |c|$.The Fourier series expansion of $\ln[b + c\cos(\lambda x + \beta)]$,

$$\ln[b + c\cos(\lambda x + \beta)] = 2\ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2}\right) - 2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}}\right)^k \cos(\lambda kx + \beta k) \tag{3}$$

for all $x \in R$.

Proof. Let $p = \frac{1}{2}(\sqrt{b+c} + \sqrt{b-c})$, $q = \frac{1}{2}(\sqrt{b+c} - \sqrt{b-c})$,then the derivative of $\ln[b + c\cos(\lambda x + \beta)]$,

$$\frac{d}{dx}\ln[b + c\cos(\lambda x + \beta)]$$

$$= \frac{-c\lambda\sin(\lambda x + \beta)}{b + c\cos(\lambda x + \beta)}$$

$$= \frac{-c\lambda\sin(\lambda x + \beta)}{p^2 + 2pq\cos(\lambda x + \beta) + q^2}$$

$$= \frac{\dfrac{-c\lambda\sin(\lambda x + \beta)}{q^2}}{\left(\dfrac{p}{q} + \cos(\lambda x + \beta)\right)^2 + \sin^2(\lambda x + \beta)}$$

$$= \text{Im}\left[\frac{\dfrac{c\lambda}{q^2}\left(\dfrac{p}{q} + \cos(\lambda x + \beta) - i\sin(\lambda x + \beta)\right)}{\left(\dfrac{p}{q} + \cos(\lambda x + \beta) + i\sin(\lambda x + \beta)\right)\left(\dfrac{p}{q} + \cos(\lambda x + \beta) - i\sin(\lambda x + \beta)\right)}\right]$$

$$= \frac{c\lambda}{pq}\cdot\text{Im}\left(\frac{1}{1 + \dfrac{qz}{p}}\right) \quad (\text{where } z = e^{i(\lambda x + \beta)})$$

$$= 2\lambda\,\text{Im}\left[\sum_{k=0}^{\infty}\left(-\frac{qz}{p}\right)^k\right] \quad \left(\text{because } \left|-\frac{qz}{p}\right| = \left|\frac{q}{p}\right| < 1\text{, we can use geometric series}\right)$$

$$= 2\lambda\sum_{k=1}^{\infty}\left(-\frac{q}{p}\right)^k \sin(\lambda kx + \beta k) \tag{4}$$

Therefore,

$$\ln[b + c\cos(\lambda x + \beta)] = -2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{q}{p}\right)^k \cos(\lambda kx + \beta k) + K \tag{5}$$

where K is a constant.

Taking $x=-\dfrac{\beta}{\lambda}$ into both sides of (5), we obtain

$$\ln(b+c)=-2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{q}{p}\right)^{k}+K \tag{6}$$

It follows that

$$\begin{aligned}K&=\ln(b+c)+2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{q}{p}\right)^{k}\\&=\ln(b+c)-2\ln\left(1+\frac{q}{p}\right)\quad\text{(by formula (i))}\\&=2\ln p\end{aligned} \tag{7}$$

Hence,

$$\begin{aligned}&\ln[b+c\cos(\lambda x+\beta)]\\&=2\ln p-2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{q}{p}\right)^{k}\cos(\lambda kx+\beta k)\\&=2\ln\left(\frac{\sqrt{b+c}+\sqrt{b-c}}{2}\right)-2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{\sqrt{b+c}-\sqrt{b-c}}{\sqrt{b+c}+\sqrt{b-c}}\right)^{k}\cos(\lambda kx+\beta k)\end{aligned}$$ ■

The following is the first major result in this study, we determine the Fourier series expansions of any order derivatives of function (1).

Theorem 1. Suppose a,b,c,λ,β are real numbers, $\lambda\neq 0$, $b>|c|$, n is any positive integer, and let the domain of $f(x)=e^{ax}\ln[b+c\cos(\lambda x+\beta)]$ be $(-\infty,\infty)$. The n -th order derivative of $f(x)$,

$$\begin{aligned}f^{(n)}(x)&=-2e^{ax}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}\cdot\lambda^{m}\sum_{k=1}^{\infty}k^{m-1}\left(-\frac{\sqrt{b+c}-\sqrt{b-c}}{\sqrt{b+c}+\sqrt{b-c}}\right)^{k}\cos\left(\lambda kx+\beta k+\frac{m\pi}{2}\right)\\&\quad+2a^{n}\ln\left(\frac{\sqrt{b+c}+\sqrt{b-c}}{2}\right)e^{ax}\end{aligned} \tag{8}$$

for all $x\in R$.

Proof. $f^{(n)}(x)$

$$=\frac{d^{n}}{dx^{n}}e^{ax}\ln[b+c\cos(\lambda x+\beta)]$$

$$= \sum_{m=0}^{n} \binom{n}{m} (e^{ax})^{(n-m)} (\ln[b + c\cos(\lambda x + \beta)])^{(m)} \quad \text{(by Leibniz differential rule)}$$

$$= \sum_{m=1}^{n} \binom{n}{m} a^{n-m} \cdot e^{ax} \left[-2\lambda^m \cdot \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{m\pi}{2} \right) \right]$$

$$+ a^n e^{ax} \left[2\ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) - 2\sum_{k=1}^{\infty} \frac{1}{k} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos(\lambda kx + \beta k) \right]$$

(by Lemma A and differentiation term by term theorem)

$$= -2e^{ax} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} \cdot \lambda^m \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{m\pi}{2} \right)$$

$$+ 2a^n \ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) e^{ax}$$

for all $x \in R$ ■

In Theorem 1, if replacing β by $\beta - \frac{\pi}{2}$, then we immediately obtain the Fourier series expansions of any order derivatives of function (2).

Theorem 2. If the assumptions are the same as Theorem 1, and the domain of $g(x) = e^{ax} \ln[b + c\sin(\lambda x + \beta)]$ is $(-\infty, \infty)$. The n -th order derivative of $g(x)$,

$$g^{(n)}(x) = -2e^{ax} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} \cdot \lambda^m \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{(m-k)\pi}{2} \right)$$

$$+ 2a^n \ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) e^{ax} \quad (9)$$

for all $x \in R$.

3. Examples

Next, aimed at the differential problem of the two types of functions in this study, we provide two functions and use Theorems 1, 2 to determine the Fourier series expansions of their any order derivatives and evaluate some of their higher order derivative values practically. In addition, we use Maple to calculate the approximations of these higher order derivative values and their solutions for verifying our answers.

Example 1. Suppose the domain of the function

$$f(x) = e^{2x} \ln\left[5 - 4\cos\left(3x - \frac{5\pi}{6} \right) \right] \quad (10)$$

is $(-\infty, \infty)$.

Then by Theorem 1, we obtain the n -th order derivative of $f(x)$,

$$f^{(n)}(x) = -2e^{2x}\sum_{m=0}^{n}\binom{n}{m}2^{n-m}\cdot 3^m \sum_{k=1}^{\infty}k^{m-1}\left(\frac{1}{2}\right)^k \cos\left(3kx-\frac{5k\pi}{6}+\frac{m\pi}{2}\right)+2^{n+1}\ln 2\cdot e^{2x} \quad (11)$$

for all $x \in R$.

Therefore, we obtain the 12-th order derivative value of $f(x)$ at $x=\frac{\pi}{2}$,

$$f^{(12)}\left(\frac{\pi}{2}\right) = -2e^{\pi}\sum_{m=0}^{12}\binom{12}{m}2^{12-m}\cdot 3^m \sum_{k=1}^{\infty}k^{m-1}\left(\frac{1}{2}\right)^k \cos\left(\frac{2k\pi}{3}+\frac{m\pi}{2}\right)+2^{13}\ln 2\cdot e^{\pi} \quad (12)$$

In the following, we use Maple to verify the correctness of (12).

```
>f:=x->exp(2*x)*ln(5-4*cos(3*x-5*Pi/6));
```

$$f := x \to e^{2x}\ln\left(5-4\cos\left(3x-\frac{5}{6}\pi\right)\right)$$

```
>evalf((D@@12)(f)(Pi/2),24);
```

$$5.91492341995799798910 7\cdot 10^9$$

```
>evalf(-2*exp(Pi)*sum(12!/(m!*(12-m)!)*2^(12-m)*3^m*sum(k^(m-1)*(1/2)^k*cos(2*k*Pi/3+m*Pi/2),k=1..infinity),m=0..12)+2^13*ln(2)*exp(Pi),22);
```

$$5.914923419957997989129\cdot 10^9$$

Example 2. Let the domain of the function

$$g(x)=e^{-3x}\ln\left[10+8\sin\left(4x+\frac{2\pi}{3}\right)\right] \quad (13)$$

be $(-\infty,\infty)$.

By Theorem 2, we can evaluate the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = -2e^{-3x}\sum_{m=0}^{n}\binom{n}{m}(-3)^{n-m}\cdot 4^m \sum_{k=1}^{\infty}k^{m-1}\left(-\frac{1}{2}\right)^k \cos\left(4kx+\frac{k\pi}{6}+\frac{m\pi}{2}\right)+2(-3)^n\cdot\ln(2\sqrt{2})\cdot e^{-3x} \quad (14)$$

for all $x \in R$.

Therefore, we obtain the 9-th order derivative value of $g(x)$ at $x=-\frac{5\pi}{6}$,

$$g^{(9)}\left(-\frac{5\pi}{6}\right)$$

$$= -2e^{5\pi/2}\sum_{m=0}^{9}\binom{9}{m}(-3)^{9-m}\cdot 4^m \sum_{k=1}^{\infty}k^{m-1}\left(-\frac{1}{2}\right)^k \cos\left(\frac{5k\pi}{6}+\frac{m\pi}{2}\right)+2(-3)^9\cdot\ln(2\sqrt{2})\cdot e^{5\pi/2} \quad (15)$$

Using Maple to verify the correctness of (15) as follows:

>g:=x->exp(-3*x)*ln(10+8*sin(4*x+2*Pi/3));

$$g := x \rightarrow e^{-3x} \ln\left(10 + 8 \sin\left(4x + \frac{2}{3}\pi\right)\right)$$

>evalf((D@@9)(g)(-5*Pi/6),24);

$$1.07193719414228756252776 \cdot 10^{14}$$

>evalf(-2*exp(5*Pi/2)*sum(9!/(m!*(9-m)!)*(-3)^(9-m)*4^m*sum(k^(m-1)*(-1/2)^k*cos(5*k*Pi/6+m*Pi/2), k=1..infinity),m=0..9)+2*(-3)^9*ln(2*sqrt(2))*exp(5*Pi/2),24);

$$1.07193719414228756252766 \cdot 10^{14} - 0.\,\mathrm{I}$$

The above answer obtained by Maple appears I ($=\sqrt{-1}$), it is because Maple calculates by using special functions built in. The imaginary part is zero, so can be ignored.

4. Conclusion

As mentioned, the differentiation term by term theorem and the Leibniz differential rule play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving, we can even use Maple to design some types of differential problems, and try to find the methods to solve them. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

References

[1] C. T. J. Dodson and E. A. Gonzalez, *Experiments in Mathematics Using Maple*, New York: Springer-Verlag, 1995.
[2] M. L. Abell and J. P. Braselton, *Maple by Example*, 3rd ed., New York: Elsevier Academic Press, 2005.
[3] J. S. Robertson, *Engineering Mathematics with Maple*, New York: McGraw-Hill, 1996.
[4] F. Garvan, *The Maple Book*, London: Chapman & Hall/CRC, 2001.
[5] R. J. Stroeker and J. F. Kaashoek, *Discovering Mathematics with Maple : An Interactive Exploration for Mathematicians, Engineers and Econometricians*, Basel: Birkhauser Verlag, 1999.
[6] D. Richards, *Advanced Mathematical Methods with Maple*, New York: Cambridge University Press, 2002.
[7] C. Tocci and S. G. Adams, *Applied Maple for Engineers and Scientists*, Boston: Artech House, 1996.
[8] C.-H. Yu, "Application of Maple on solving the differential problem of rational functions," *Applied Mechanics and Materials*, 2013, in press.
[9] C. -H. Yu, " A study on the differential problems using Maple," *International Journal of Computer Science and Mobile Computing* , vol. 2, issue. 7, pp. 7-12, July 2013.
[10] C. -H. Yu, "Application of Maple on solving some differential problems," *Proceedings of IIE Asian Conference 2013*, National Taiwan University of Science and Technology, Taiwan, vol. 1, pp. 585-592, July 2013.
[11] C.-H. Yu, "A study on some differential problems with Maple," *Proceedings of 6th IEEE/International Conference on Advanced Infocomm Technology*, National United University, Taiwan, no. 00291, July 2013.
[12] C. -H. Yu, " Evaluating the derivatives of trigonometric functions with Maple," *International Journal of Research in Computer Applications and Robotics*, vol. 1, issue. 4, pp. 23-28, July 2013.

[13] C. -H. Yu, " The differential problem of two types of exponential functions, " *Journal of Nan Jeon*, vol. 16, in press.
[14] C. -H. Yu, "Application of Maple: taking the differential problem of rational functions as an example," *Proceedings of 2012 Optoelectronics Communication Engineering Workshop*, National Kaohsiung University of Applied Sciences, Taiwan, pp. 271-274, October 2012.
[15] C. -H. Yu, " A study on the differential problem of some trigonometric functions, " *Journal of Jen-Teh*, vol. 10, pp. 27-34.
[16] C. -H. Yu," The differential problem of two types of functions," *International Journal of Computer Science and Mobile Computing*, vol. 2, issue. 7, pp. 137-145, July 2013.
[17] C. -H. Yu, "Application of Maple on the differential problem of hyperbolic functions," *Proceedings of International Conference on Safety & Security Management and Engineering Technology 2012*, WuFeng University, Taiwan, pp. 481-484, June 2012.
[18] C. -H. Yu, " The differential problem of four types of functions," *Journal of Kang-Ning*, vol. 14, in press.
[19] C. -H. Yu," Using Maple to evaluate the derivatives of some functions," *International Journal of Research in Computer Applications and Robotics*, vol. 1, issue. 4, pp. 23-31, July 2013.
[20] C. -H. Yu, "A study on the differential problem," *International Journal of Research in Aeronautical and Mechanical Engineering*, vol. 1, issue. 3, pp. 52-57, July 2013.
[21] C. -H. Yu, "Application of Maple: taking the evaluation of higher order derivative values of some type of rational functions as an example," *Proceedings of 2012 Digital Life Technology Seminar*, National Yunlin University of Science and Technology, Taiwan, pp.150-153, August 2012.
[22] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Boston: Addison-Wesley, p230, 1975.
[23] R. Courant and F. John, *Introduction to Calculus and Analysis*, New York: Springer-Verlag, vol. 1, p203, 1989.

