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The Differential Problem of Some Functions

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Abstract

This paper uses the mathematical software Maple for the auxiliary tool to study the differential problem of two types of functions. We can obtain the infinite series forms of any order derivatives of these functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of calculating their higher order derivative values. On the other hand, we propose two functions to evaluate their any order derivatives, and calculate some of their higher order derivative values practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords: derivatives; infinite series forms; differentiation term by term theorem; Leibniz differential rule; Maple

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Mozart, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to [1]-[7].

In calculus courses, determining the n-th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x=c$, in general, needs to go through two procedures: firstly finding the n-th order derivative $f^{(n)}(x)$ of $f(x)$, and secondly taking $x=c$ into $f^{(n)}(x)$. These two procedures will make us face with increasingly complex calculations when calculating higher order derivative values of a function (i.e. n is large), Therefore, to obtain the answers by manual calculations is not easy. In this paper, we mainly study the differential problems of the following two types of functions

$$f(x) = x^a \ln(1 + x^b) \tag{1}$$

$$g(x) = x^a \tan^{-1} x^b \tag{2}$$

where a, b are real numbers, $b > 0$. We can obtain the infinite series forms of any order derivatives of these two types of functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of evaluating their higher order derivatives values ; these are the major results of this paper (i.e., Theorems 1, 2). As for the study of related differential problems can refer to [8]-[15]. On the other hand, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and two formulas used in this study.

Notation.

Suppose r is any real number, n is any positive integer. Define $(r)_n = r(r-1)\cdots(r-n+1)$, and $(r)_0 = 1$.

Formulas.

(i) $\ln(1+y) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}y^k$, where $-1 < y \le 1$.

(ii) $\tan^{-1} y = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}y^{2k-1}$, where $-1 \le y \le 1$.

Next, we introduce two important theorems used in this paper.

Differentiation term by term theorem ([16]).

If, for all non-negative integer k , the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b) , (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a,b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) . Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

Leibniz differential rule ([17]). Assume n is a positive integer. If $f(x), g(x)$ are functions such that their k -th derivatives $f^{(k)}(x), g^{(k)}(x)$ exist for all $k = 1,\ldots,n$. Then the n -th derivative of product function $f(x)g(x)$,

$$(fg)^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Firstly, we determine the infinite series forms of any order derivatives of function (1).

Theorem 1. Suppose a, b are real numbers, $b > 0$, n is any positive integer, and let the domain of $f(x) = x^a \ln(1 + x^b)$ be $\left\{x \in R \middle| x^a, x^b exist, x > -1, x \neq 0\right\}$.

Case (1). If $-1 < x \leq 1$, then the n-th order derivative of $f(x)$,

$$f^{(n)}(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a + bk)_n \cdot x^{a+bk-n} \tag{3}$$

Case (2). If $x > 1$, then

$$f^{(n)}(x) = x^{a-n} \cdot \left[b \cdot \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} (k-1)! (a)_{n-k} + b \cdot (a)_n \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a - bk)_n x^{-bk} \right] \tag{4}$$

Proof. Case (1). If $-1 < x \leq 1$, then $-1 < x^b \leq 1$ and

$$\begin{aligned} f(x) &= x^a \ln(1 + x^b) \\ &= x^a \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (x^b)^k \qquad \text{(by formula (i))} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} x^{a+bk} \end{aligned} \tag{5}$$

Therefore, by differentiation term by term theorem, differentiating n-times with respect to x on both sides of (5), we obtain

$$f^{(n)}(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a + bk)_n \cdot x^{a+bk-n}$$

Case (2). If $x > 1$, then $x^b > 1$ (because $b > 0$), and

$$\begin{aligned} f(x) &= x^a \ln(1 + x^b) \\ &= x^a \ln\left[x^b \left(1 + \frac{1}{x^b} \right) \right] \\ &= x^a \ln(x^b) + x^a \ln\left(1 + \frac{1}{x^b} \right) \\ &= bx^a \ln x + x^a \cdot \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (x^{-b})^k \qquad \text{(by formula (i))} \\ &= bx^a \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} x^{a-bk} \end{aligned} \tag{6}$$

Thus, by differentiation term by term theorem and Leibniz differential rule, differentiating n-times with respect to x on both sides of (6), we have

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$$f^{(n)}(x)$$

$$= b \cdot \sum_{k=0}^{n} \binom{n}{k} (x^a)^{(n-k)} (\ln x)^{(k)} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a-bk)_n \cdot x^{a-bk-n}$$

$$= b \cdot (a)_n \cdot x^{a-n} \ln x + b \cdot \sum_{k=1}^{n} \binom{n}{k} (a)_{n-k} \cdot x^{a-n+k} \frac{(-1)^{k-1}(k-1)!}{x^k} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a-bk)_n \cdot x^{a-bk-n}$$

$$= x^{a-n} \cdot \left[b \cdot \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1}(k-1)!\,(a)_{n-k} + b \cdot (a)_n \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (a-bk)_n x^{-bk} \right] \quad \blacksquare$$

Next, we determine the infinite series form of any order derivatives of function (2).

Theorem 2. If the assumptions are the same as Theorem 1, and let the domain of $g(x) = x^a \tan^{-1} x^b$ be $\{x \in R \mid x^a, x^b \text{exist}, x \geq -1, x \neq 0\}$.

Case (1). If $-1 \leq x \leq 1$, then the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} (a-b+2bk)_n \cdot x^{a-b+2bk-n} \tag{7}$$

Case (2). If $x > 1$, then

$$g^{(n)}(x) = x^{a-n} \left[\frac{\pi}{2} (a)_n - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} (a+b-2bk)_n x^{b-2bk} \right] \tag{8}$$

Proof. Case (1). If $-1 \leq x \leq 1$, then $-1 \leq x^b \leq 1$ (because $b > 0$) and

$$g(x) = x^a \tan^{-1} x^b$$

$$= x^a \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} (x^b)^{2k-1} \qquad \text{(by formula (ii))}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} x^{a-b+2bk} \tag{9}$$

Therefore, by differentiation term by term theorem, differentiating n -times with respect to x on both sides of (9), we obtain

$$g^{(n)}(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} (a-b+2bk)_n \cdot x^{a-b+2bk-n}$$

Case (2). If $x > 1$, then $x^b > 1$ (because $b > 0$), and

$$g(x) = x^a \tan^{-1} x^b$$

$$= x^a\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x^b}\right)\right]$$

$$= \frac{\pi}{2}x^a - x^a\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}\left(\frac{1}{x^b}\right)^{2k-1} \qquad \text{(by formula (ii))}$$

$$= \frac{\pi}{2}x^a - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}x^{a+b-2bk} \tag{10}$$

Hence, by differentiation term by term theorem, differentiating n -times with respect to x on both sides of (10), we have

$$g^{(n)}(x) = \frac{\pi}{2}(a)_n x^{a-n} - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}(a+b-2bk)_n x^{a+b-2bk-n}$$

$$= x^{a-n}\left[\frac{\pi}{2}(a)_n - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}(a+b-2bk)_n x^{b-2bk}\right] \qquad \blacksquare$$

3. Examples

Regarding the differential problem of the two types of functions in this study, we provide two functions and use Theorems 1, 2 to determine the infinite series forms of their any order derivatives and evaluate some of their higher order derivative values practically. In addition, we employ Maple to calculate the approximations of these higher order derivative values and their infinite series forms for verifying our answers.

Example 1. Suppose the domain of the function

$$f(x) = x^{13/7}\ln(1+x^{6/5}) \tag{11}$$

is $\{x \in R \mid x > -1, x \neq 0\}$.

Case (1). If $-1 < x \le 1$, by Case (1) of Theorem 1, we can determine the n -th order derivative of $f(x)$,

$$f^{(n)}(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\left(\frac{13}{7}+\frac{6}{5}k\right)_n \cdot x^{13/7+6k/5-n} \tag{12}$$

Therefore, we obtain the 11-th order derivative value of $f(x)$ at $x = 1/4$,

$$f^{(11)}\left(\frac{1}{4}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\left(\frac{13}{7}+\frac{6}{5}k\right)_{11} \cdot \left(\frac{1}{4}\right)^{6k/5-64/7} \tag{13}$$

In the following, we use Maple to verify the correctness of (13).

>f:=x->x^(13/7)*ln(1+x^(6/5));

$$f := x \to x^{13/7}\ln\left(1+x^{6/5}\right)$$

>evalf((D@@11)(f)(1/4),14);

$$-1.2571009134935 \cdot 10^{8}$$

>evalf(sum((-1)^(k-1)/k*product(13/7+6*k/5-j,j=0..10)*(1/4)^(6*k/5-64/7),k=1..infinity),14);

$$-1.2571009134935 \cdot 10^{8}$$

Case (2). If $x > 1$, by Case (2) of Theorem 1, the n-th order derivative of $f(x)$,

$$f^{(n)}(x)$$

$$= x^{13/7-n} \cdot \left[\frac{6}{5} \cdot \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} (k-1)! (13/7)_{n-k} + \frac{6}{5} \cdot (13/7)_n \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (13/7 - 6k/5)_n \, x^{-6k/5} \right] \tag{14}$$

Thus, we can determine the 13-th order derivative value of $f(x)$ at $x = 5$,

$$f^{(13)}(5)$$

$$= 5^{-78/7} \cdot \left[\frac{6}{5} \cdot \sum_{k=1}^{13} \binom{13}{k} (-1)^{k-1} (k-1)! (13/7)_{13-k} + \frac{6}{5} \cdot (13/7)_{13} \ln 5 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (13/7 - 6k/5)_{13} \cdot 5^{-6k/5} \right] \tag{15}$$

We also use Maple to verify the correctness of (15) .

>evalf((D@@13)(f)(5),18);

0.1474083829324055

>evalf(5^(-78/7)*6/5*sum(13!/(k!*(13-k)!)*(-1)^(k-1)*(k-1)!*product(13/7-j,j=0..(12-k)),k=1..13)+5^(-78/7)*6/5*product(13/7-p,p=0..12)*ln(5)+5^(-78/7)*sum((-1)^(k-1)/k*product(13/7-6*k/5-j,j=0..12)*5^(-6*k/5),k=1..infinity),18);

0.147408382932405639

Example 2. Suppose the domain of the function

$$g(x) = x^{9/4} \tan^{-1}(x^{8/3}) \tag{16}$$

is $\{x \in R \,|\, x > 0\}$.

Case (1). If $0 < x \le 1$, by Case (1) of Theorem 2, we obtain the n-th order derivative of $g(x)$,

$$g^{(n)}(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} \left(-\frac{5}{12} + \frac{16}{3} k \right)_n \cdot x^{-5/12 + 16k/3 - n} \tag{17}$$

Therefore, we can evaluate the 14-th order derivative value of $g(x)$ at $x = 1/2$,

$$g^{(14)}\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} \left(-\frac{5}{12} + \frac{16}{3} k \right)_{14} \cdot \left(\frac{1}{2}\right)^{16k/3 - 173/12} \tag{18}$$

Using Maple to verify the correctness of (18) as follows:

```
>g:=x->x^(9/4)*arctan(x^(8/3));
```

$$g := x \to x^{9/4} \arctan\left(x^{8/3}\right)$$

```
>evalf((D@@14)(g)(1/2),18);
```

$$1.52073359301369628 \cdot 10^{12}$$

```
>evalf(sum((-1)^(k-1)/(2*k-1)*product(-5/12+16*k/3-j,j=0..13)*(1/2)^(16*k/3-173/12),k=1..infinity),18);
```

$$1.52073359301369604 \cdot 10^{12}$$

Case (2). If $x > 1$, by Case (2) of Theorem 2, we can determine the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = x^{9/4-n}\left[\frac{\pi}{2}(9/4)_n - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}\left(\frac{59}{12}-\frac{16}{3}k\right)_n x^{8/3-16k/3}\right] \tag{19}$$

Thus, we obtain the 9-th order derivative value of $g(x)$ at $x = 8$,

$$g^{(9)}(8) = 8^{-27/4}\left[\frac{\pi}{2}(9/4)_9 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}\left(\frac{59}{12}-\frac{16}{3}k\right)_9 8^{8/3-16k/3}\right] \tag{20}$$

We also use Maple to verify the correctness of (20) .

```
>evalf((D@@9)(g)(8),22);
```

$$0.0004646884381751081$$

```
>evalf(8^(-27/4)*Pi/2*product(9/4-j,j=0..8)-8^(-27/4)*sum((-1)^(k-1)/(2*k-1)*product(59/12-16*k/3-j,j=0..8)
*8^(8/3-16*k/3),k=1..infinity),18);
```

$$0.000464688438175108005 - 1.96775344653668938 \cdot 10^{-22}\, \mathrm{I}$$

The above answer obtained by Maple appears I $(=\sqrt{-1}\,)$, it is because Maple calculates by using special functions built in. But the imaginary part is very small, so can be ignored.

4. Conclusion

From the above discussion, we know the differentiation term by term theorem and the Leibniz differential rule play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavour to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving; we can even use Maple to design some types of differential problems, and try to find the methods to solve them. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

References

[1] D. Richards, *Advanced Mathematical Methods with Maple*, New York: Cambridge University Press, 2002.
[2] F. Garvan, *The Maple Book*, London: Chapman & Hall/CRC, 2001.
[3] C. T. J. Dodson and E. A. Gonzalez, *Experiments in Mathematics Using Maple*, New York: Springer-Verlag, 1995.
[4] R. J. Stroeker and J. F. Kaashoek, *Discovering Mathematics with Maple : An Interactive Exploration for Mathematicians, Engineers and Econometricians*, Basel: Birkhauser Verlag, 1999.
[5] M. L. Abell and J. P. Braselton, *Maple by Example*, 3rd ed., New York: Elsevier Academic Press, 2005.
[6] C. Tocci and S. G. Adams, *Applied Maple for Engineers and Scientists*, Boston: Artech House, 1996.
[7] J. S. Robertson, *Engineering Mathematics with Maple*, New York: McGraw-Hill, 1996.
[8] C. -H. Yu, "Application of Maple on solving some differential problems," *Proceedings of IIE Asian Conference 2013*, No. 0108, National Taiwan University of Science and Technology, Taiwan, July 2013, in press.
[9] C.-H. Yu, "A study on some differential problems with Maple," *Proceedings of 6th IEEE/International Conference on Advanced Infocomm Technology*, National United University, Taiwan, July 2013, in press.
[10] C. -H. Yu, " The differential problem of two types of exponential functions, " *Journal of Nan Jeon*, vol. 16, in press.
[11] C. -H. Yu, "Application of Maple: taking the differential problem of rational functions as an example," *Proceedings of 2012 Optoelectronics Communication Engineering Workshop*, National Kaohsiung University of Applied Sciences, Taiwan, pp. 271-274, October 2012.
[12] C. -H. Yu, " A study on the differential problem of some trigonometric functions, " *Journal of Jen-Teh*, vol. 10, pp. 27-34.
[13] C. -H. Yu, "Application of Maple on the differential problem of hyperbolic functions," *Proceedings of International Conference on Safety & Security Management and Engineering Technology 2012*, WuFeng University, Taiwan, pp. 481-484, June 2012.
[14] C. -H. Yu, "The differential problem of four types of functions," *Journal of Kang-Ning*, vol. 14, in press.
[15] C. -H. Yu, "Application of Maple: taking the evaluation of higher order derivative values of some type of rational functions as an example," *Proceedings of 2012 Digital Life Technology Seminar*, National Yunlin University of Science and Technology, Taiwan, pp.150-153, August 2012.
[16] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Boston: Addison-Wesley, p230, 1975.
[17] R. Courant and F. John, *Introduction to Calculus and Analysis*, New York: Springer-Verlag, vol. 1, p203, 1989.

